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U-transfer schemes and dynamical systems in n-person TU-games

Juan Carlos Cesco *Ana Lucía Calí*

Instituto de Matemática Aplicada San Luis (CONICET-U.N.San Luis)
(jcesco@unsl.edu.ar)
Departamento de Matemática (U.N.San Luis)
Chacabuco y Pedernera, 5700 San Luis, Argentina
(acali@unsl.edu.ar)

Abstract

In this paper we define a non-continuous discrete dynamical system related to a transfer scheme designed originally to approximate imputations in the core of balanced games. We show that the dynamical system may have either periodic point of period 1 (fixed points) or periodic points with period greater than one, but not both. Moreover, the fixed points of the dynamical system characterize the core of a balanced game. On the other side, periodic points of period greater than one are associated with certain class of cycles of pre-imputations that can appear in non-balanced games (maximal U-cycles). For monotonic non-balanced 3-person games we describe completely the set of periodic points and their associated (forward) stable sets.

1 Introduction.

In a recent paper (Cesco (2003)) a characterization of non-balanced games in terms of the existence of certain cycles of pre-imputations (fundamental cycles) has been proved. Later it was shown that, for some class of TU-games (games with transferable utilities), the characterization theorem can still be obtained using narrow classes of cycles, U-cycles and maximal U-cycles (Cesco, Calí (2003)). The latter has the advantage of being a class of computable cycles using an algorithm developed to reach points in the core of a balanced TU-game (Cesco (1998)). Along with this algorithm we associate a discrete dynamical system. The aim of this paper is to study the set of periodic points of this dynamical system and some properties of their associated stable sets. Concerning this issue, we obtain a full characterization in the framework of monotonic non-balanced 3-person games. The main result of this paper concerning these games (theorem 6) indicates that every imputation is forward asymptotic (section 2). This fact has important consequences from a computational point of view. On one hand, it allows us to give a precise notion of global convergence of the imputations generated by the algorithm developed in Cesco (1998) to maximal U-cycles. On the other, the positive results obtained here encourage us and provide some insight to deal with more general classes of games.

The paper is organized as follows. Preliminaries and some notation are set forth in the next section. In section 3 we define cycles of pre-imputation and close it showing that the existence of a U-cycles or maximal U-cycles in a TU-game implies the non-balancedness of it (theorem 1) for a sub-class of n-person games which generalizes 3-person games. We close this section with some concepts related to dynamic systems. There we prove that the dynamical system defined in this note may have either fixed points or periodic points of period greater than one, but not both. In the last section we study the case of non-balanced monotonic 3-person games. In this framework we characterize the stable set of imputations associated with periodic imputations of period greater than one.

2 Preliminaries.

A TU-game is a pair (N, v) where $N = \{1, 2, ..., n\}$ represents the set of players and v the characteristic function. We assume that v is a real valued function defined on the family of subsets of $N, \mathcal{P}(N)$ satisfying $v(N) = 1$ and $v(\{i\}) = 0$ for each $i \in N$. The elements in $\mathcal{P}(N)$ are called coalitions. The game is called monotonic if $v(S) \geq v(T)$ whenever $S \supseteq T$.

The set of pre-imputations is defined by $E = \{x = (x_1, \ldots, x_n) \in R^n : \sum_{i \in N} x_i = 1\}$ and the set of imputations by $A = \{x \in E : x_i \geq 0 \text{ for all } i \in N\}$.

Given a coalition $S \in \mathcal{P}(N)$ and a pre-imputation x, the excess of the coalition S with respect to x is defined by $e(S, x) = v(S) - x(S)$, where $x(S) = \sum_{i \in S} x_i$ if $S \neq \Phi$ and 0 otherwise. The excess of a coalition S represents the aggregate gain (or loss, if negative) to its members if they depart from an agreement that yields

x in order to form their own coalition. The core of a game (N, v) is defined by $C = \{x \in E : e(S, x) \leq 0$ for all $S \in P(N)\}$.

The core of a game may be an empty set. The Shapley-Bondareva's theorem (Bondareva (1963), Shapley (1968)) characterizes the sub class of TU-games with non-empty core. A central role is played by balanced families of coalitions. A family of non-empty coalitions $\mathcal{B} \subseteq \mathcal{P}(N)$ is called a *balanced* if there exists a set of positive real numbers $(\lambda_S)_{S\in\mathcal{B}}$ satisfying $\sum\limits_{\substack{S\in\mathcal{B}\\ S\ni i}} \lambda_S = 1$, for all $i \in N$. The numbers $(\lambda_S)_{S\in\mathcal{B}}$ are called the balancing weights for $\mathcal{B}$. $\mathcal{B}$ is minimal balanced if there is no proper balanced subfamily of it. In this case, the set of balanced weights is unique. Equivalently, if $\chi_S \in R^n$ denotes the characteristic vector defined by $(\chi_S)_i = 1$ if $i \in S$ and 0 if $i \in N\backslash S$, the family $\mathcal{B}$ is balanced if there exists a family of positive balancing weights $(\lambda_S)_{S\in\mathcal{B}}$, such that

$$\sum_{S\in\mathcal{B}} \lambda_S.\chi_S = \chi_N$$

(Cesco (2003)). A well-known result establishes that

$$\sum_{S\in\mathcal{B}} \lambda_S.x\,(S) = x\,(N)$$

for all balanced family of coalitions. A game (N, v) is called balanced if

$$\sum_{S\in\mathcal{B}} \lambda_S.v(S) \leq v(N) \tag{1}$$

for all balanced family $\mathcal{B}$ with balancing weights $(\lambda_S)_{S\in\mathcal{B}}$. The Shapley- Bondareva's theorem states that a game (N, v) has non-empty core if and only if it is balanced. An *objectionable* family is a balanced family not satisfying (1).

In what follows, the notion of U-transfer will play a central role. Given $x \in E$ and a proper coalition S, we say that y results from x by the *U-transfer* from $N\backslash S$ to S (shortly, y is a U-transfer from x) if

$$y = x + e(S, x).\beta_S$$

Here $\beta_S = \frac{\chi_S}{|S|} - \frac{\chi_{N\backslash S}}{|N\backslash S|}$ if S is a proper coalition and the zero vector of R^n otherwise. $|S|$ indicates the number of players in S. The vector β_S describes a transfer of one unit of utility from the members of $N\backslash S$ to the members of S. The U-transfer called maximal if $e(S, x) \geq e(T, x)$ for all $T \in \mathcal{P}(N)$.

$>$From now on, we will consider the coalitions in $\mathcal{P}(N)$ indexed somehow from 0 to $2^n - 1$, with $S_0 = \Phi$ and $S_{2^n-1} = N$.

Given $x \in E$, let $\varphi(x) = \{i : e(S_i, x) \geq e(S_j, x)$ for all $j = 0, ..., 2^n - 1\}$. Let $g : E \to E$ the function defined by

$$g(x) = x + e(S_i, x).\beta_{S_i} \text{ with } i \in \varphi(x) \text{ and } i \leq j \text{ for all } j \in \varphi(x) \tag{2}$$

The function g defines, by iteration, a discrete dynamical system.
Remark 1 The following example shows that, in general, g is not a continuous function. Let (N, v) be a 3-person game with characteristic given by

$$\begin{aligned} v(N) &= v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = 1 \\ v(S) &= 0 \text{ otherwise} \end{aligned}$$

Here we consider $S_1 = \{1,2\}, S_2 = \{2,3\}, S_3 = \{1,3\}$. Let $x = (\frac{1}{2}, 0, \frac{1}{2})$. Then, $\varphi(x) = \{1,2\}$, and

$$\begin{aligned} g(x) &= x + e(S_1, x).\beta_{S_1} \\ &= (\frac{1}{2}, 0, \frac{1}{2}) + \frac{1}{2}.(\frac{1}{2}, \frac{1}{2}, -1) \\ &= (\frac{3}{4}, \frac{1}{4}, 0) \end{aligned}$$

If ε is positive and small enough, and $x^\varepsilon = (\frac{1}{2}+\varepsilon, 0, \frac{1}{2}-\varepsilon)$, then $\varphi(x^\varepsilon) = \{2\}$ and $g(x^\varepsilon) = (0, \frac{1}{4} + \frac{1}{2}\varepsilon, \frac{3}{4} - \frac{1}{2}\varepsilon)$ which is not close to $g(x)$ for all ε close to zero.

For completeness, we list below some basic definitions from the theory of dynamical systems. For further references we refer the reader to Devaney (1989).

Given a discrete dynamic system defined by a function $g : X \to X$, the *orbit* (forward orbit) of $x \in X$ is the sequence of points $x, g(x), g^2(x), \ldots$ A point x is called a *fixed* point for g if $g(x) = x$. A point x is a *periodic* point of *period* n if $g^n(x) = x$. The least positive n for which $g^n(x) = x$ is called the *prime* period of x. The set of all periodic points of (not necessarily prime) period n will be denoted by $Per_n(g)$, and the set of all fixed points by $Fix(g)$. Given a periodic point x, the set $P = \{y \in X : y = g^k(x) \text{ for some } k \geq 1\}$ is called a periodic orbit. Sometimes we will use the use the notation $P(x)$ to stress the fact that the periodic orbit P is associated to the periodic point x.

Periodic point may exhibits some stability properties. Let $\bar{x}$ be a periodic point of period n. A point x is forward asymptotic to $\bar{x}$ if $\lim\limits_{i\to\infty} g^{i.n}(x) = \bar{x}$. The stable set of $\bar{x}$, denoted by $W^s(\bar{x})$, consists of all points forward asymptotic to $\bar{x}$.

The goal of dynamical systems is to understand the nature of all orbits. Generally this is an impossible task. However, related to the dynamical system defined by (2), we can give a complete description of its orbits whenever the associated game is 3-person game.

3 Cycles of pre-imputations.

In this section we introduce two kind of cycles of pre-imputations and state, without proof several, results proved in Cesco and Aguirre (2002), Cesco (2003).
Definition 1 A $U-$*cycle* $\mathbf{c}$ in a TU-game (N, v) is a finite sequence of pre-imputations $(x^k)_{k=1}^m, m > 1$, such that there exist associated sequences $(S_k)_{k=1}^m$

of non-empty, proper coalitions of N (not necessarily all different) satisfying the neighbouring transfer properties

$$x^{k+1} = x^k + e(S_k, x^k).\beta_{S_k} \text{ for all } k = 1, \ldots, m$$

and

$$x^{m+1} = x^1$$

as well.

We refer to the numbers $(e(S_k, x^k))_{k=1}^m$ as the *transfer amounts.*

Given a cycle $\mathbf{c} = (x^k)_{k=1}^m$, we denote the family $(S_k)_{k=1}^m$ by supp($\mathbf{c}$), and by $X(\mathbf{c}) = \{x \in E : x = x^r$ for some pre-imputation x^r in the cycle $\mathbf{c}\}$.

In Cesco (2003), theorem 1 we prove that supp($\mathbf{c}$)= $(S_k)_{k=1}^m$ is a balanced family of coalitions. This result was stated for a more general class of cycles (*fundamental cycles)* than U-cycles.

Remark 3 The existence of fundamental cycles in a TU-game is strongly related to the non-existence of points in the core of the game. The results proved in Cesco (2003) (theorems 3 and 9) allow us to state that a TU-game (N, v) is balanced (i.e. with non-empty core) if and only if there do not exist fundamental cycles in (N, v).

In Cesco, Calí (2003) we prove that, in some cases, the class of fundamental cycles can be narrowed and still obtain a similar equivalence theorem. There we state the following result.

Theorem 1 Let (N, v) be a TU-game with characteristic function given by

$$\begin{aligned} v(S_1) &= \mu_n, v(S_k) = \mu_{k-1} \text{ for all } k = 2, ..., n \\ v(N) &= 1, \text{ and } v(S) = 0 \text{ otherwise} \end{aligned}$$

Then the following statements are equivalent

(i) The game is balanced.

(ii) There does not exist a U-cycle in the game.

If the game is monotonic, then (i) is equivalent to

(iii) There does not exist a maximal U-cycle in the game.

Remark 4 The advantage of maximal U-cycles over fundamental cycles is that the former are computable. In Cesco (1998) we introduced the notion of *maximal transfer schemes* in TU-games. A maximal transfer scheme is a sequence $(x^k)_{k=1}^{\infty}$of pre-imputations such that x^{k+1} is a maximal U-transfer from x^k for all $k = 1, 2, ...$ There we proved that if a maximal transfer scheme converges, the core of the game is non-empty (corollary 3.2). Besides, we implemented an algorithm for computing maximal transfer schemes. When applied to non-balanced games, it always 'converged' to maximal U-cycles.

The following simple result provides some insight about periodic points.

Proposition 2 Let (N, v) be an n-person TU-game, and g a dynamical system defined by (2). Then, *i)* if x is a periodic point of period $k, k \geq 2$ then $(x, g(x), ..., g^{k-1}(x))$ is a maximal U-cycle.

ii) x is a fixed point for g if and only if $x \in C$.

Proof The proof of *i)* is straightforward. To prove *ii)*,we first assume that x is a fixed point for g. Then

$$x = x + e(S, x).\beta_S$$

for some $S \in \mathcal{P}(N)$ satisfying $e(S, x) \geq e(T, x)$ for all $T \in \mathcal{P}(N)$. Thus, $e(S, x).\beta_S = 0$. If $S \neq \Phi, N$, $\beta_S \neq 0$ and we conclude that $e(S, x) = 0$. But the same is true if $S = \Phi$ or $S = N$. In any case we get that the maximum excess with respect to x is 0, and this implies that $x \in C$. Conversely, if $x \in C$, $\max_{S \in \mathcal{P}(N)} \{e(S, x)\} = 0$. So $g(x) = x$ for any dynamical system defined through (2). ■

Because of this result and Theorem 3 in Cesco (2003) we have the following

Corollary 3 Let (N, v) be an n-person TU-game, and g a dynamical system defined by (2). Then, g can have periodic points of period 1 (fixed points) or periodic points of period $k > 1$, but not both. Moreover, if the game (N, v) is balanced, $C = Fix(g)$.

For general non-balanced games we do not know if there exist maximal U-cycles. We have positive results for three person games (Cesco, Aguirre (2002) and for a subclass of n-person games which behave quite similar to 3-person games (Cesco, Calí (2003)).

In the next section we give a complete description of the periodic orbits in non-balanced 3-person games.

4 3-person games

In this section we address to the characterization of the set of periodic imputations and their stable sets in the framework of 3-person games. The following results were proved in Cesco, Aguirre (2002).

Lemma 4 Let G be 3-person game with the following characteristic function

$$v(N) = 1, v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = \mu \qquad (3)$$

and $v(S) = 0$ otherwise. If $\mu > \frac{2}{3}, \mathbf{c} = \{x^1, x^2, x^3\}$ where

$$x^1 = (\frac{1}{3}, 1 - \mu, \mu - \frac{1}{3}), x^2 = (\mu - \frac{1}{3}, \frac{1}{3}, 1 - \mu), x^3 = (1 - \mu, \mu - \frac{1}{3}, \frac{1}{3})$$

is a maximal U-cycle.

Remark 5 The condition $\mu > \frac{2}{3}$ is necessary and sufficient to guarantee the non-balancedness of the game.

Lemma 5 Let G be 3-person game with the following characteristic function

$$\begin{aligned} v(N) &= 1, v(\{1, 2\}) = \mu_3, v(\{1, 3\}) = \mu_2, v(\{2, 3\}) = \mu_1 && (4) \\ v(S) &= 0 \text{ otherwise} && (5) \end{aligned}$$

Then $\mathbf{c} = \{x^1, x^2, x^3\}$ where

$$\begin{aligned} x^1 &= (\frac{1}{3} + t_1, 1 - \mu + t_2, \mu - \frac{1}{3} + t_3) \\ x^2 &= (\mu - \frac{1}{3} + t_1, \frac{1}{3} + t_2, 1 - \mu + t_3) \\ x^3 &= (1 - \mu + t_1, \mu - \frac{1}{3} + t_2, \frac{1}{3} + t_3) \end{aligned}$$

Here

$$\mu = \frac{1}{3}(\mu_1 + \mu_2 + \mu_3) \tag{6}$$

and

$$\begin{aligned} t = \; & (\frac{1}{2}(\mu_2 + \mu_3 - \mu_1) - \frac{1}{2}\mu, \\ & \frac{1}{2}(\mu_1 + \mu_3 - \mu_2) - \frac{1}{2}\mu, \\ & \frac{1}{2}(\mu_1 + \mu_2 - \mu_3) - \frac{1}{2}\mu) \end{aligned} \tag{7}$$

is a maximal U-cycle provided $\mathcal{B} = \{\{1,2\},\{1,3\},\{2,3\}\}$ is an objectionable family.

Now we state a connection between cycles given in lemma 4 and lemma 5.

Definition 4 Let $G = (N, v)$ a 3-person game with characteristic function given by (4-5). We define the w-equivalence of G as the 3-person game $\tilde{G}$ defined by

$$\begin{aligned} \tilde{v}(N) &= 1, \tilde{v}(\{1,2\}) = \mu, \tilde{v}(\{1,3\}) = \mu, \tilde{v}(\{2,3\}) = \mu \\ \tilde{v}(S) &= 0 \text{ otherwise} \end{aligned}$$

with μ given by (6).

We point out that the w-equivalence left invariant the worth of the minimal balanced family of $\mathcal{B} = \{\{1,2\},\{1,3\},\{2,3\}\}$.

Remark 6 Let $G = (N, v)$ be a 3-person monotonic game with empty core. We point out that its w-equivalence also has empty core. Moreover, if $\tilde{\mathbf{c}} = \{\tilde{x}^1, \tilde{x}^2, \tilde{x}^3\}$ is the cycle given in lemma 4, then, there exists a vector $\mathbf{t} = \mathbf{t}_G = (t_1, t_2, t_3)$ such that $\mathbf{c} = \tilde{\mathbf{c}} + \mathbf{t}$ is a cycle in G. In fact, the vector $\mathbf{t}$ is given by (7). For an interpretation of the vector $\mathbf{t}$, we refer the reader to Cesco, Aguirre (2002). This implies that there exists a one to one correspondence between the maximal U-cycles in the game with characteristic function given by (3) and the general non-balanced 3-person game with characteristic function given by $(4-5)$. Moreover, an appropriate dilatation/contraction transformation with respect to the barycentre imputation $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ establishes a one to one correspondence between the maximal U-cycles of any two games with characteristic function given by (3) (Cesco, Calí (2003b, corollary 14)).

In what follows, we restrict ourselves to work with 3-person games whose characteristic function is given by (3) with $\mu = 1$. Because of remark 6, this will no represent any loss of generality.

Theorem 5 Let $G = (N, v)$ be a game with characteristic function given by (3) with $\mu = 1$. Let $\tilde{\mathbf{c}} = \{\tilde{x}^1, \tilde{x}^2, \tilde{x}^3\}$ be the cycle

$$\tilde{x}^1 = (\frac{1}{3}, 0, \frac{2}{3}), \tilde{x}^2 = (\frac{2}{3}, \frac{1}{3}, 0), \tilde{x}^3 = (0, \frac{2}{3}, \frac{1}{3})$$

Then, if $x^1 = (x_1^1, x_2^1, x_3^1)$ is an imputation satisfying $x_3^1 > x_1^1 > x_2^1 = 0$, then, the maximal U-sequence $\{x^i\}_{i\geq 1}$ starting in x^1 has the following properties:

$$a.- \quad \begin{array}{l} \max_S e(S, x^{1+3i}) = e(\{1,2\}, x^{1+3i}) = x_3^{1+3i} \\ \max_S e(S, x^{2+3i}) = e(\{2,3\}, x^{2+3i}) = x_1^{2+3i} \\ \max_S e(S, x^{3+3i}) = e(\{1,3\}, x^{3+3i}) = x_2^{3+3i} \end{array}$$

for all $i \geq 1$.

$$b.- \quad \begin{array}{l} \lim_i x^{1+3i} = \tilde{x}^1 \\ \lim_i x^{2+3i} = \tilde{x}^2 \\ \lim_i x^{3+3i} = \tilde{x}^3 \end{array}$$

Proof Under the hypothesis we have that $e(\{1,3\}, x^1) = 0$. Let $S_1 = \{1,2\}, S_2 = \{2,3\}, S_3 = \{1,3\}$. We first show that if a pre-imputation $x \neq \tilde{x}^i$ satisfies $e(S_i, x) = 0$ for some $i = 1, 2, 3$, then

$$\left\|y - \tilde{x}^{i+1}\right\|_2 < \left\|x - \tilde{x}^i\right\|_2$$

where $y = x + e(S_i, x)\beta_{S_i}$. Here the indexes $i+1$ are considered mod(3). It has been proved that β_S is orthogonal to the affine submanifold $E(S) = \{x \in E : e(S, x) = 0\}$ for all non trivial coalition S (Cesco (1998), lemma 2.2). Therefore, since $y, \tilde{x}^{i+1}$ are the orthogonal projections onto $E(S_i)$ of $x, \tilde{x}^i$ respectively, we get that

$$\left\|y - \tilde{x}^{i+1}\right\|_2 \leq \left\|x - \tilde{x}^i\right\|_2$$

Our claim follows by noting that $\|y - x\|_2 = e(S_i, x)\left\|\beta_{S_i}\right\|_2$ is different from $e(S_i, x)\left\|\beta_{S_i}\right\|_2 = \left\|\tilde{x}^{i+1} - \tilde{x}^i\right\|_2$, so, the strict inequality must hold in the above relation. Moreover, it holds that

$$\left\|y - \tilde{x}^{i+1}\right\|_2 = \cos(\frac{\pi}{3})\left\|x - \tilde{x}^i\right\|_2 \tag{8}$$

Second, we point out that $e(S_1, x^1) = 1 - (x_1^1 + x_2^1) = x_3^1$. Similarly, $e(S_2, x^1) = x_1^1$ and $e(S_3, x^1) = x_2^1$. So the maximum excess is reached at S_1, and its value is x_3^1. Therefore, $x^2 = x^1 + x_3^1.(\frac{1}{2}, \frac{1}{2}, -1)$. It is easy to show that $x_1^2 > x_2^2 > x_3^2 = 0$. A simple calculation indicates that the maximum excess for x^2 is reached at S_2. With the same arguments already employed, we prove that $x^3 = x^2 + x_1^2.(-1, \frac{1}{2}, \frac{1}{2})$ satisfies $x_2^3 > x_3^3 > x_1^3 = 0$, and that its maximum excess is reached at S_3. One more step indicates x^{1+3} satisfies $x_3^{1+3} > x_1^{1+3} > x_2^{1+3} = 0$ which are the same conditions satisfied by x^1. The validity of $a.-$ is now obtained by using an inductive argument.

To prove $b.-$, we take into account (8). Therefore

$$\left\|x^4 - \tilde{x}^1\right\|_2 \quad = \quad \cos(\frac{\pi}{3})\left\|x^3 - \tilde{x}^3\right\|_2$$

$$
\begin{aligned}
&= (\cos(\frac{\pi}{3}))^2 \left\| x^2 - \tilde{x}^2 \right\|_2 \\
&= (\cos(\frac{\pi}{3}))^3 \left\| x^1 - \tilde{x}^1 \right\|_2
\end{aligned}
$$

This generalizes to

$$\left\| x^{1+3i} - \tilde{x}^1 \right\|_2 = (\cos(\frac{\pi}{3}))^{3i} \left\| x^1 - \tilde{x}^1 \right\|_2$$

for all $i \geq 1$. Thus, we get that $\lim_i x^{1+3i} = \tilde{x}^1$. The other cases are proved similarly. ■

We close the note with some remarks.

Remark 7 If the imputation x^1 satisfies $x_1^1 > x_3^1 > x_2^1 = 0$, then the convergence is to the maximal U-cycle $\bar{\mathbf{c}} = \{\bar{x}^1, \bar{x}^2, \bar{x}^3\}$,where

$$\bar{x}^1 = (\frac{2}{3}, 0, \frac{1}{3}), \bar{x}^2 = (0, \frac{1}{3}, \frac{2}{3}), \bar{x}^3 = (\frac{1}{3}, \frac{2}{3}, 0)$$

Theorem 6 Let $G = (N, v)$ be a game with characteristic function given by (3) with $\mu = 1$, and let g be the dynamical system defined by (2). Then, $W^s(\tilde{x}^1) \cap A = \{x \in A : x_3 \geq x_1 \geq x_2\}$. Similarly, $W^s(\bar{x}^1) \cap A = \{x \in A : x_1 > x_3 \geq x_2\}$.

Proof Let an imputation $x = (x_1, x_2, x_3)$ satisfy $x_3 \geq x_1 \geq x_2$. Since

$$
\begin{aligned}
e(S_1, x) &= 1 - (x_1 + x_2) \\
&= x_3
\end{aligned}
$$

and similarly, $e(S_2, x) = x_1, e(S_3, x) = x_2$, we get that $e(S_1, x) \geq e(S_2, x) \geq e(S_3, x)$. Besides, $e(S_1, x) > e(S_2, x)$ whenever $x_3 > x_1$. In any case, according to the definition of g,

$$
\begin{aligned}
x^2 &= g(x) \\
&= x + e(S_1, x).\beta_{S_1}
\end{aligned}
$$

satisfies $e(S_1, x^2) = 0$. Similar calculations shows that $x^4 = g^3(x)$ satisfies the hypothesis of theorem 5. Therefore, the sequence $x^{1+3i} = g^{3i}(x)$ converges to $\tilde{x}^1$. Thus, we have shown that $W^s(\tilde{x}^1) \cap A \supseteq \{x \in A : x_3 \geq x_1 \geq x_2\}$. We can use a similar argument to show that an imputation x satisfying $x_1 > x_3 \geq x_2$ belongs to $W^s(\bar{x}^1) \cap A$. Moreover, with the same technique and taking into account the definition of g,we can prove that

$$
\begin{aligned}
W^s(\tilde{x}^2) \cap A &\supseteq \{x \in A : x_1 \geq x_2 > x_3\} \\
W^s(\bar{x}^3) \cap A &\supseteq \{x \in A : x_2 > x_1 > x_3\} \\
W^s(\tilde{x}^3) \cap A &\supseteq \{x \in A : x_2 > x_3 \geq x_1\} \\
W^s(\bar{x}^2) \cap A &\supseteq \{x \in A : x_3 \geq x_2 > x_1\}
\end{aligned}
$$

Since the sets on the right hand side of these inclusions form a partition of A, we conclude that all the equalities must hold. This concludes our proof. ■

We point out that the above result implies that the only periodic imputations in the 3-person game studied here are those in the maximal U-cycles $\tilde{\mathbf{c}}$ and $\bar{\mathbf{c}}$. All of them have period 3. Because of remark 6 there are only six periodic imputations in all monotonic non-balanced 3-person game G. Let us call them $\tilde{x}^k_G$ and $\bar{x}^k_G, k = 1,2,3$. The stable sets associated with these six periodic imputations can be obtained as follows. Let $\mathbf{t} = \mathbf{t}_G$ be the vector mentioned in Remark 6. Then $W^s(\tilde{x}^k_G) \cap A = (W^s(\tilde{x}^k) + \mathbf{t}_G) \cap A$, and $W^s(\bar{x}^k_G) \cap A = (W^s(\bar{x}^k) + \mathbf{t}_G) \cap A, k = 1,2,3$.

It is interesting to add that the transformations mentioned in remark 6 provide indeed, a one to one correspondence between the maximal U-sequences of a monotonic non-balanced game with characteristic function given by $(4-5)$ and those of a game with characteristic function given by (3) with $\mu = 1$. Therefore, there exists a one to one correspondence between the orbits in a general monotonic non-balanced 3-person game and those of the game studied in theorem 6.

5 Conclusions

The main result presented in this paper provides a complete characterization for the set of periodic points and its corresponding stable sets of monotonic non-balanced 3-person games. However, the proof given here, which is based on theorem 5, depends strongly on the number of players in the game. It would be desirable to obtain similar characterization results for more general games. In Cesco, Calí (2003b) we address to the case of n-person games in which the only permissible coalitions are those of cardinality $1, n-1$ and n. These games are a quite good generalization of the 3-person games studied in this note. Nothing has been done regarding other classes of games.

On the other side, the imputations in a periodic orbit, and some coalitions associated to them could be used to model some processes of coalition formation, a central point in cooperative game theory. However, in general games, we do not expect to get closed forms to describe the periodic orbits in the case they exist. It is an open issue to characterize subsets of forward asymptotic imputations without knowing, in advance, the periodic imputation to which they converge. In this case, the algorithm developed in Cesco (1998) could be used to find out periodic points. Some computational experience shows that many games are globally stable in the sense that every imputation is forward asymptotic. This numerical evidence and the results obtained in this note and in Cesco, Calí (2003b) indicate that this line of work is worth to be explored further.

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